1  Motivation

A recent study sponsored by the US Government concluded that enterprise-wide ’’... modeling and simulation are emerging as key technologies to support manufacturing in the 21st century, and no other technology offers more potential than modeling and simulation for improving products, perfecting processes, reducing design-to-manufacturing cycle time, and reducing product realization costs...’’.

The Finite Element Method (FEM), which is the most popular modeling approach, is a very successful story. It is a way that engineers invented to solve equations of mechanics to find displacements and stresses in structures. The history of FEM begins in 1915 with B. Galerkin who used 2-3 sophisticated trial functions. In 40’s A. Hrennikoff and R. Courant had two different conceptions but the common key point to use many simple trial functions. First developments of the method are associated with the following names: J. Argyris, M. Turner, R. Clough (who coined the terminology finite element method) and O. Zienkiewicz. Mathematicians contributed to understanding as well as reliability of the FEM and nowadays computers enable handling efficiently even $10^{12}$ d.o.f. but still new shape functions and algorithms are invented and tested for solids, fluids, coupled (e.g. thermo-mechanical) problems, wave propagation, chemical reactions, quantum mechanics, ...

- Finite element analysis makes a good engineer great, and a bad engineer dangerous (R.Cook).
- What you hear, you forget; what you see, you remember; what you do, you understand.
- Modeling (simulating nature) gives us insight into the world we live in.
- An engineer has to know how to assume a model, solve it on a laptop, and assess the accuracy of the results.

2  General idea of the finite element method

1. Let’s consider the Laplace PDE equation (as a model of e.g. stationary heat transfer)

$$\begin{cases}
  u(x, y) \in C^2(\Omega) : \ R^2 \supset \Omega \rightarrow R \\
  -k\Delta u = q(x, y) \quad \text{in} \ \Omega \\
  u = \hat{u} \quad \text{on} \ \partial\Omega_D \\
  k\frac{\partial u}{\partial n} = \hat{t} \quad \text{on} \ \partial\Omega_N
\end{cases} \tag{1}$$

2. Weak formulation (”virtual work principle”) of the Laplace problem:

Find continuous $u(x, y) \in H^1(\Omega) + \hat{u}$, such that $u = \hat{u}$ on $\partial\Omega_D$ and

$$\int_{\Omega} k \nabla v \cdot \nabla u \, dx \, dy = \int_{\Omega} v q \, dx \, dy + \int_{\partial\Omega_N} \hat{v} \hat{t} \, d\gamma \quad \forall v \in H^1_0 \tag{2}$$

3. FEM (Galerkin’s method with solution approximation by shape functions)

- Domain and its discretization with 2 finite elements (4 nodes)

\[ \hat{u} = 3x - 2 \]

- Selected shape functions - \( \varphi_1(x, y), \varphi_2(x, y) \)

- Continuous approximation of the solution

\[ u_h(x, y) = \alpha_1 \varphi_1(x, y) + \alpha_2 \varphi_2(x, y) + \alpha_3 \varphi_3(x, y) + \alpha_4 \varphi_4(x, y) \]

\( \alpha_1, \alpha_2, \ldots \alpha_N \) - unknown parameters - degrees of freedom (d.o.f.)

- Galerkin’s method

\[ v \in \{ \varphi_1, \varphi_2, \varphi_3, \varphi_4 \} \]

4 algebraic equations - ”virtual work” for ”virtual” displacements \( \varphi_1, \varphi_2, \varphi_3, \varphi_4 \)

Let’s assume the following notations:

\[ (\varphi_i, \varphi_j)_{m} = \alpha_j \int_{e_m} k \nabla \varphi_i \cdot \nabla \varphi_j \, dx \, dy \]

\[ (\varphi_i)_{m} = \int_{\Omega} \varphi_i \, q \, dx \, dy + \int_{\partial \Omega_N \cap \partial e_m} \varphi_i \, t \, d\gamma \]

\[
\begin{align*}
(\varphi_1, \varphi_1)_1 &+ (\varphi_1, \varphi_1)_2 + (\varphi_1, \varphi_1)_3 + (\varphi_1, \varphi_1)_4 + (\varphi_1, \varphi_1)_5 + (\varphi_1, \varphi_1)_6 = (\varphi_1)_1 + (\varphi_1)_2 \\
(\varphi_2, \varphi_1)_1 &+ (\varphi_2, \varphi_1)_2 + (\varphi_2, \varphi_1)_3 + (\varphi_2, \varphi_1)_4 + (\varphi_2, \varphi_1)_5 + (\varphi_2, \varphi_1)_6 = (\varphi_2)_1 + (\varphi_2)_2 \\
(\varphi_3, \varphi_1)_1 &+ (\varphi_3, \varphi_1)_2 + (\varphi_3, \varphi_1)_3 + (\varphi_3, \varphi_1)_4 + (\varphi_3, \varphi_1)_5 + (\varphi_3, \varphi_1)_6 = (\varphi_3)_1 + (\varphi_3)_2 \\
(\varphi_4, \varphi_1)_1 &+ (\varphi_4, \varphi_1)_2 + (\varphi_4, \varphi_1)_3 + (\varphi_4, \varphi_1)_4 + (\varphi_4, \varphi_1)_5 + (\varphi_4, \varphi_1)_6 = (\varphi_4)_1 + (\varphi_4)_2 \\
\end{align*}
\]

Entries in gray are equal to 0. Entries in red and green are integrals over elements 1 and 2 respectively.
• $m$-th element (stiffness) matrix and (load) vector
  \[ K_{ij}^m = \int_{e_m} k \nabla \varphi_i \cdot \nabla \varphi_j \, dx \, dy, \quad P_i^m = \int_{e_m} \varphi_i q \, dx \, dy + \int_{\partial \Omega \cap \partial e_m} \varphi_i \hat{t} \, d\gamma \]

• Accounting for Dirichlet (essential, kinematic) boundary conditions
  e.g. if $\hat{u} = 3x - 2$ is given at segment AD it implies that $\alpha_1 = -2$, $\alpha_2 = 1$
  Therefore equations 1 and 2 are not needed.

• Postprocessing (with possible assessment of the result quality)
  e.g. for element 1, d.o.f. ($\alpha_1$, $\alpha_2$, $\alpha_4$) are already known, thus in that element one may
  evaluate

  approximation of solution: $u_h(x, y) = \alpha_1 \varphi_1(x, y) + \alpha_2 \varphi_2(x, y) + \alpha_4 \varphi_4(x, y)$
  and flux: $q = \alpha_1 \nabla \varphi_1(x, y) + \alpha_2 \nabla \varphi_2(x, y) + \alpha_4 \nabla \varphi_4(x, y)$

4. Homework

Consider problem (1) in the domain shown in this section (item 3).

• Write the formulas for FEM approximation of the solution and its flux knowing that $\alpha_3 = 2$, $\alpha_4 = -2/3$.

• Prove that such an approximation is continuous and satisfies the Dirichlet boundary condition. Evaluate $\alpha_1, \alpha_2$ for $\hat{u} = x^2$

• Check accuracy of satisfying the Neumann b.c. if $\hat{t}_1 = -1$, $\hat{t}_2 = 0$, $\hat{t}_3 = 1 - x$

3 Recap of the finite element method algorithm

1. An exemplary problem - 1D bar (axial deformations)

\[ q(x, t) - \text{continuous} \]
  
  Newton’s principle: $\frac{d}{dt} p = F \quad \forall \omega \subset \Omega, t \in [0, T]$
  \[
  \mathcal{H}
  \]

  Hamilton’s principle: $\delta \int_0^T (K - U + W) \, dt = 0$

  K - kinetic energy, U - potential energy, W - work done by loading
2. Strong formulation: Find $u(x,t) \in C^2$, such that

$$\begin{cases}
A\ddot{u} + Ac\dot{u} - AEu'' = q(x,t) \quad \forall x \in (0,l), \forall t \in [0,T] \\
u(0,t) = 0 \quad \forall t \in [0,T] \\
AEu'(l,t)n(l) = P(t) \quad \forall t \in [0,T] \\
u(x,0) = u_0(x) \quad \forall x \in [0,l] \\
\dot{u}(x,0) = v_0(x) \quad \forall x \in [0,l]
\end{cases}$$ (4)

3. Weak formulation: Find $u(x,t) \in H_1$, such that $u(0,t) = 0 \\forall t$ and

$$\int_0^l vA\ddot{u} \, dx + \int_0^l vAc \dot{u} \, dx + \int_0^l vAEu' \, dx = \int_0^l vq \, dx + v(l)P \quad \forall v \in V_0, \forall t \in [0,T]$$ (5)

4. FEM (Galerkin’s method and solution approximation by shape functions)

Let’s assume that $\dot{u} = 0 \\forall t$. Then $u = u(x)$.

- Discretization (finite elements)

- Shape functions (e.g. on element 3, using local enumeration)

  element by element algorithm

  $\hat{\varphi}_1(x) = \frac{(x-x_2)}{(x_1-x_2)}$, $\hat{\varphi}_2(x) = \frac{(x-x_1)}{(x_2-x_1)}$, $\hat{\varphi}_3(x) = (x-x_1)(x-x_2)$

  remarks: $\hat{\varphi}_1(x) + \hat{\varphi}_2(x) = 1$, $\hat{\varphi}_1(x) + \hat{\varphi}_2(x) + \hat{\varphi}_3(x) \neq 1$

  position of the third node is neither specified nor used

- Approximation of a solution (over element 3)

  $u_h(x) = \hat{\alpha}_1 \hat{\varphi}_1(x) + \hat{\alpha}_2 \hat{\varphi}_2(x) + \hat{\alpha}_3 \hat{\varphi}_3(x)$, d.o.f. $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$ - degrees of freedom (d.o.f.)

  $\hat{\alpha}_1 = u_h(\hat{x}_1)$, $\hat{\alpha}_2 = u_h(\hat{x}_2)$

- Approximation of the geometry (on element 3)

  $x = \hat{x}_1 \hat{\varphi}_1(x) + \hat{x}_2 \hat{\varphi}_2(x)$

- Galerkin’s method

$$\int_0^l \varphi_i' AEu_h' \, dx = \int_0^l \varphi_i q \, dx + \varphi_i(l)P \quad \forall \varphi_i, \ i = 1, ..., N$$ (6)
• Element stiffness matrix and load vector
\[ K_{ij}^e = \int_e \dot{\phi}_i' AE \dot{\phi}_j' \, dx, \quad P_i = \int_e \dot{\phi}_i q \, dx \]

or in a matrix form (for a 2 dof element)
\[ K^e = \int_e B^T D B \, dx, \quad B = [\dot{\phi}_1 \dot{\phi}_2 \dot{\phi}_3], \quad D = AE \]
\[ P^e = \int_e N^T q \, dx, \quad N = [\dot{\phi}_1 \dot{\phi}_2 \dot{\phi}_3] \]

• Assembling

• Accounting for kinematic conditions
• SLE solution; \( Ku = P (+F) \); \( F \) - nodal forces for bar structures only
• Postprocessing with assesment of the result quality
  e.g. for element 3, d.o.f. \((\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)\) are already known, thus

  continuous approximation of axial displacement: \( u_h(x) = \hat{\alpha}_1 \dot{\phi}_1(x) + \hat{\alpha}_2 \dot{\phi}_2(x) + \hat{\alpha}_3 \dot{\phi}_3(x) \)

  discontinuous approximation of axial stress: \( \sigma_h(x) = E \frac{du_h}{dx} = \hat{\alpha}_1 \dot{\phi}_1'(x) + \hat{\alpha}_2 \dot{\phi}_2'(x) + \hat{\alpha}_3 \dot{\phi}_3'(x) \)

  discontinuous approximation of axial force: \( S_h(x) = A \sigma_h(x) \)

  improved (in fact exact) axial nodal values of axial forces: \( F^e = K^e u^e - P^e \)

5. Homework

Consider the following 1D bar problem
Find \( u(x) \in H_1([a,b]) \), such that \( u(a) = u(b) = 0 \) and

\[ \int_a^b u' A E u' \, dx = \int_a^b v q_0 (l-x) \, dx \quad \forall v \in V_0, \quad q_0 \in \mathbb{R}, \quad l = b - a \]  

\[ (7) \]

• Calculate the matrix and vector for the three node finite element \([2,3]\) with hierarchical shape functions up to the second order.
• Assemble the matrix and vector assuming that the global node numbers are 1,3,2 (left, mid, right).
• Write the formulas used for FEM approximation of axial displacement and axial force over that element.
• Discretize a bar represented by segment \([1,3]\) with 2 second order hierarchical elements.
• Calculate FEM approximation of the axial force and the improved axial forces.
• Check the global equilibrium equation.
4 FEM analysis of mechanical vibrations

Vibrations - incredible common phenomenon. In certain cases they are a positive thing (speech, music) and sometimes a negative one (noise of braking pads, vibration of buildings, bridges,...). Engineers are involved in both making and suppressing vibrations.

1. Types of the related problems
   - Eigen vibrations.
   - Response to time dependent loading (periodic or random).
   - Wave propagation.

2. Single DOF system - without external excitation

\[
m \ddot{x} + c \dot{x} + kx = 0 \quad \forall t \in (0, \tau)
\]
\[
x(0) = x_0 \quad \dot{x}(0) = v_0
\]

\[\omega = \sqrt{\frac{k}{m}}\] - undamped natural (angular) frequency

\[\delta = \frac{c}{2m}, \quad \zeta = \frac{\delta}{\omega}\] - damping ratio

\[\alpha_{1,2} = \omega(-\zeta \pm \sqrt{\zeta^2 - 1}) \in C, \quad x = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}, \quad e^{i\theta} = \cos(\theta) + i \sin(\theta)
\]

- \(\zeta = 0 \rightarrow \ddot{x} + \frac{k}{m}x = 0 \rightarrow x = A \sin(\omega t - \varphi) \text{ or } x = A \sin(\omega t) \pm B \cos(\omega t)
\]

- \(0 < \zeta \leq 1 \rightarrow x = Ae^{-\zeta \omega t} \cos(\omega_d t - \varphi), \omega_d = \sqrt{\omega^2 - \zeta^2} \) - damped natural frequency

- \(\zeta > 1 \rightarrow x = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}, \alpha_1, \alpha_2, \omega \in R
\)
3. Axial vibrations of elastic bars

Wave propagation

\[ u(x,t) \]

Longitudinal wave

Particles vibrate in the direction of wave propagation

4. Axial free vibrations of elastic bars

Standing (stationary, fixed in space) waves - combination (interference) of two waves moving in opposite directions, each having the same amplitude and frequency. When waves are superimposed, their energies are either added together or cancelled out. The PDE of motion reduces in this case to the following eigenproblem (typically, the initial conditions are not considered).

\[
\begin{cases}
\rho \ddot{u} - Eu'' = 0 & \forall x \in (0, l), \forall t \in [0, \tau] \\
u(0, t) = 0 & \forall t \in [0, \tau] \\
\nu'(l, t) = 0 & \forall t \in [0, \tau]
\end{cases}
\]

(8)

Separation of variables, \( u(x,t) = U(x)V(t) \), results in two ODEs

\[
\begin{cases}
\ddot{V} + \omega_n^2 V = 0 \\
U'' + k_n^2 U = 0
\end{cases}
\]

(9)

Thus, the general, nontrivial solution is of the form

\[ u(x,t) = \sum_{n=1}^{\infty} [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)] \left[ (C_n \cos(k_n x) + D_n \sin(k_n x)) \right], \quad k_n = \omega_n \sqrt{\frac{\rho}{E}} - \text{wave numbers}
\]

(10)

\[ \omega_n = 2\pi f_n, \quad \omega_n T_n = 2\pi, \quad k_n \lambda_n = 2\pi, \quad \frac{\lambda_n}{T_n} = \frac{\omega_n}{k_n} = \sqrt{\frac{E}{\rho}} - \text{sound (phase) speed}
\]

The Dirichlet b.c. implies that \( C_n = 0 \)

The Neumann b.c. implies the characteristic equation

\[ \cos(k_n l) = 0 \quad \Rightarrow \quad k_n l = (2n - 1)\frac{\pi}{2}, \quad n = 1, 2, 3, \ldots \]

(11)

and one obtains the following angular frequency spectrum and mode shapes

\[ \omega_n = (2n - 1)\frac{\pi}{2l} \sqrt{\frac{E}{\rho}}, \quad u_n(x) = \beta \sin((2n - 1)\frac{\pi}{2l} x) \]

(12)
5. Finite element analysis of free vibrations

- Approximation: \( u_h(x,t) = \alpha_1(t)\varphi_1(x) + \alpha_2(t)\varphi_2(x) + \ldots + \alpha_N(t)\varphi_N(x) \)

- In general, there are 3 element matrices
  
  \[ M^{(e)} = \int_e N^T A \rho N \, dx \]
  \[ C^{(e)} = \int_e N^T A_c N \, dx \]
  \[ K^{(e)} = \int_e B^T A E B \, dx \]

- After assembling these matrices and accounting for essential b.c. one obtains the following system of second order ODEs
  
  \[
  \begin{cases}
  \ddot{M}u + \dot{C}u + Ku = f(t) \quad \forall t \in (0,T) \\
  u(t=0) = u_0 \\
  \dot{u}(t=0) = v_0
  \end{cases}
  \]

- Free undamped vibrations \( \Rightarrow \) eigenproblem
  
  Find \( u \neq 0 \) such that \( \ddot{M}u + Ku = 0 \quad \forall t \in (0,T) \)

  Since solution has the following form
  
  \( u = U \sin(\omega t + \psi) \)

  one obtains
  
  \((-\omega^2 \dot{M}U + \dot{K}U) \sin(\omega t + \psi) = 0 \quad \forall t \in (0,T)\)

  that reduces to the following generalized algebraic eigenproblem
  
  \[
  \begin{cases}
  \dot{K}U = \omega^2 \dot{M}U \\
  U^T \dot{M}U = \mu_0, \quad \mu_0[J \cdot s^2] \text{ is an arbitrary constant, typically 1}
  \end{cases}
  \]

  which, in turn, enables computation of approximate mode shapes \((U_1, U_2, \ldots)\) and corresponding natural frequencies \((\omega_1, \omega_2, \ldots)\)

- Due to the symmetry of \( \dot{M} \) and \( \dot{K} \) the eigen modes are orthogonal, i.e.
  
  \[
  U_i^T \dot{M}U_j = 0 \quad \text{for } i \neq j
  \]
  \[
  U_i^T \dot{K}U_j = 0 \quad \text{for } i \neq j
  \]
  \[
  U_k^T \dot{M}U_k = \mu_0 \quad \text{(maximum kinetic energy divided by } \omega^2)\]
  \[
  U_k^T \dot{K}U_k = \omega_k^2 \mu_0 \quad \text{(maximum potential energy)}
  \]

6. Numerical integration in time

   For the initial value problem \( \frac{dy}{dt} = f, \ y(t_0) = y_0 \)

   one introduces discretization in time represented by time step \( \Delta t \).

   Thus, \( y(t_0 + \Delta t) = y_0 + \int_{t_0}^{t_0 + \Delta t} f \, dt \)

   The last integral is evaluated after assuming an approximation for integrand \( f \), that is equivalent to assuming \textit{certain approximation of } y \textit{ with respect to time.}
7. Types of numerical methods for integration in time
   • Single-step and multi-step.
   • Explicit and implicit.
   • Conditionally or unconditionally stable.

8. Newmark’s method (for $\beta = 0.5$, $\gamma = 1$)
   Without damping $\ddot{M}\ddot{u} + \dot{K}u = f(t) \ \forall t \in (0, T)$
   • For given: $u_0$, $v_0$, $Ma_0 = f_0 - Ku_0$
   • $(M + \frac{1}{2}\triangle t^2 K)a_{k+1} = f_{k+1} - \dot{K}(u_k + \triangle tv_k)$
   • $u_{k+1} = u_k + \triangle tv_k + \frac{1}{2}\triangle t^2 a_{k+1}$
   • $v_{k+1} = v_k + \triangle t a_{k+1}$

9. Modal approximation (for harmonic response)
   • $u = z_1(t)U_1 + z_2(t)U_2 + \ldots + z_N(t)U_N$, $U_i$ - eigen vectors
   • $\ddot{z}_i + 2\zeta_i\omega_i \dot{z}_i + \omega_i^2 z_i = F_i$, $\zeta_i = \frac{\alpha + \beta\omega_i^2}{2\omega_i}$ \quad $\forall i = 1, ..., N$ (decoupled system of ODEs)

10. Fourier analysis
   • Fourier series for a periodic function ($T = 2l$):
   \[
   FS(f)(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l}
   \]
   where
   \[
   a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx, \quad a_0 = \frac{1}{2l} \int_{-l}^{l} f(x) \, dx
   \]
   \[
   b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx, \quad C_n = \frac{1}{2l} \int_{-l}^{l} f(x)e^{-in\pi x/l} \, dx
   \]
   • $\frac{n\pi}{l} = \omega_n$, $f_n = \frac{n}{T}$ - sequence of angular frequencies (infinite spectrum)
   • A function may be represented both in time and frequency domains
   TIME domain - $f(x)$
   FREQUENCY domain - $|C_n|/\omega_n$ ($\omega_n = \frac{n\pi}{T}$)
• For non-periodic function \((T = \infty)\)

Fourier transform to frequency domain

\[
\begin{align*}
a(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos(\omega x) \, dx, \\
b(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin(\omega x) \, dx, \\
C(\omega) &= \int_{-\infty}^{\infty} f(x)e^{-2\pi i\omega x} \, dx
\end{align*}
\]

Inverse transform

\[
f(x) = \int_{-\infty}^{\infty} [a \cos(\omega x) + b \sin(\omega x)] \, d\omega = \int_{-\infty}^{\infty} C(\omega)e^{2\pi i\omega x} \, d\omega
\]

TIME domain - \(f(x)\) 

FREQUENCY domain - \(C(\omega)\) \((\omega \in \mathbb{R})\)

• In practice: DFT, FFT

11. Homework

• Use 3 finite elements with linear shape functions for a bar and
  
  – compute global stiffness and mass matrices for such a discretization
  – calculate spectrum of natural frequencies and draw the corresponding mode shapes
  – verify orthogonality of the modes, calculate modal masses and stiffnesses
  – repeat the above proposed analysis for a lumped mass matrix
  – Explain differences between single and multi step methods, explicit and implicit
    schemas, stable and unstable methods.
  – Sketch graphs of \(f(t) = 2 \cos(\pi t) - \sin(\frac{\pi t}{2})\) in time and frequency domains.

• Knowing that

\[
u(x, t) = \sum_{n=1}^{\infty} \left[ A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \right] \left( C_n \cos(k_n x) + D_n \sin(k_n x) \right)
\]

is a general form of the eiegen function (exact free vibration displacements) for a bar clamped at both ends, determine the bar eigen frequencies and eigen modes.

• Use 3 finite elements with linear shape functions for the bar considered in problem 1 and
  
  – compute global stiffness and mass matrices for such a discretization
  – calculate spectrum of natural frequencies and draw the corresponding mode shapes
  – verify orthogonality of the modes, calculate modal masses and stiffnesses
  – repeat the above proposed analysis for a lumped mass matrix
5 FEM for 2D and 3D problems

1. Formulation of a steady-state thermo-mechanical problem (with neglected interior source terms)

\[ \text{Find } u \in H^1_0(\Omega) + h \text{ and } \theta \in H^1_0 + T, \text{ such that:} \]
\[
\begin{cases}
\int_{\Omega} \epsilon(v) : C \epsilon(u) \, d\Omega - \int_{\Omega} \text{tr}(\epsilon(v)) \alpha \theta \, d\Omega = \int_{\partial\Omega} v q \, ds \\
\int_{\Omega} \nabla \psi : k \nabla \theta \, d\Omega = \int_{\partial\Omega} \psi S \, ds
\end{cases} \tag{15}
\]
\[ \forall v \in H^1_0(\Omega), \forall \psi \in H^1_0(\Omega) \]

2. Element stiffness matrix and load vector \( \mathbf{N}_1 = [\varphi_1, 0, 0], \mathbf{N}_2 = [0, \varphi_1, 0], \ldots \)

\[
\begin{align*}
K_{uij} &= \int_{e} \epsilon(N_i) : C \epsilon(N_j) \, d\Omega \\
K_{ij}^{u\theta} &= \int_{e} \text{tr}(\epsilon(N_i)) \alpha \varphi_j \, d\Omega \\
K_{ij}^{\theta\theta} &= \int_{e} \nabla \varphi_i : k \nabla \varphi_j \, d\Omega \\
P_{ui} &= \int_{\partial e \cap \partial \Omega} N_i q \, ds \\
P_{i\theta} &= \int_{\partial e \cap \partial S} \varphi_i S \, ds \tag{18, 19}
\end{align*}
\]

3. Assembling element matrices (vectors) into global matrix (vector) - on the basis of d.o.f. connectivities (local with global numbering relations)

4. Exemplary 2D domain and discretization by 3 elements

5. Selected scalar \((\varphi_i)\) vertex and edge shape functions
6. Static boundary conditions (only distributed loading is used in well posed problems)
Non-zero entries of load vector (using local node numbering) for the triangular element

\[
\hat{P}^u_4 = \int_{\partial e \cap \Omega_t} [0, \frac{x}{2}][0, q(x)]^T \, ds
\] (20)

\[
\hat{P}^u_6 = \int_{\partial e \cap \Omega_t} [0, x(2 - x)][0, q(x)]^T \, ds
\] (21)

\[
\hat{P}^u_8 = \int_{\partial e \cap \Omega_t} [0, 1 - x/2][0, q(x)]^T \, ds
\] (22)

7. Kinematic boundary conditions
Both components of displacement along the part of the boundary \((x = 0, 1 \leq y \leq 3)\) are 0.
Never use pointwise kinematic boundary conditions in 2D and 3D problems of second order!

8. Postprocessing
\[
u = \sum^m_{k=1} \alpha_k N_k \text{ (continuous)} \Rightarrow \varepsilon, \sigma \text{ (discontinuous)}
\] (23)

9. A’posteriori error estimation and mesh adaptation

![Figure 1: Heterogeneous material distribution (colors represent different materials) and hp-adapted FEM mesh (colors represent order of approximation). Note the “hanging”.](image)

10. Homework
- For the following plane strain problem and discretization by 3 elements

- sketch graphs of all shape functions
- calculate load vectors for all elements
- calculate displacements at one interior point of each element for the following d.o.f.
  \([0, 0, 0, 0, 0, 0, 0.5, -1, 1, -2, 1, -2, 2, -3, 2, -2.5]10^{-2}\)
- Describe briefly the methods of a’posteriori error estimation
11. Why FEM (a Galerkin approach) is the best?

It delivers the best possible approximation \( u_h \) in the sense of the energy norm

\[
\|\|\| e \|\| = \sqrt{a(e, e)}
\]

One may prove that for symmetric positive definite bilinear forms by error orthogonality and the Schwartz inequality

\[
\|\| u - u_h \|\|^2 = a(u - u_h, u - u_h) = a(u - w_h + w_h - u_h, u - u_h) = a(u - w_h, u - u_h) \leq \|\| u - w_h \|\| \|\| u - u_h \|\|
\]

that implies

\[
\|\| u - u_h \|\| \leq \|\| u - w_h \|\| \quad \forall w_h \in V_h
\]

6 Selected other computer methods

1. XFEM - extended FEM

- A FEM version for modeling discontinuities by enrichment of selected shape functions by the Heaviside function. This way, also theoretically predicted solution behavior is built into the approximation (e.g. in vicinity of 2D crack tip by \( \sqrt{r} \sin(\frac{\theta}{2}), \sqrt{r} \cos(\frac{\theta}{2}), \ldots \)).
- Example of enrichment of selected shape function, let’s say \( \varphi_I \)

\[
u_h = \ldots + u_I \varphi_I + \ldots \rightarrow \quad u_h = \ldots + (u_I + a_I \Psi_a + b_I \Psi_b) \varphi_I + \ldots
\]

where \( \Psi_a \varphi_I, \Psi_b \varphi_I \) are additional shape functions attributed to the same node as the \( \varphi_I \) function and \( a_I, b_I \) are the corresponding additional degrees of freedom at that node.
- 1D example

![Figure 2: A 1D basis shape function (\( \varphi_I \)) and its two enrichments (\( \Psi_a \varphi_I, \Psi_b \varphi_I \)) by distance from \( a=0.75 \) and Heaviside function, i.e. \( \Psi_a = |x - 0.75|, \ \Psi_b = H(x - 0.75) \).](image-url)
2. BEM - boundary element method

- Somigliana’s identity
  \[ c u(\xi) = \int_{\partial \Omega} [u^*(x, \xi)t(x) - t^*(x, \xi)u(x)] d\gamma + \int_{\Omega} u^*(x, \xi)f(x) d\Omega \]  
  (28)

- Boundary integral equation (for \(u|_{\partial \Omega} = 0\))
  \[ \int_{\partial \Omega} u^*(x, \xi)t(x) d\gamma + \int_{\Omega} u^*(x, \xi)f(x) d\Omega = 0 \]  
  (29)

- Discretization for \(t \in L^2(\partial \Omega)\) by piecewise constant function and the collocation method result in the following SLAE
  \[ t_j \int_{\partial \gamma_j} u^*(x, \xi_i) d\gamma_x = - \int_{\Omega} u^*(x, \xi_i)f(x) d\Omega_x \quad \forall i, j = 1, 2, \ldots, N \]  
  (30)

3. Meshless, FDM - finite difference method

- MLS (moving least squares) approximation - \(\varphi(z)\)
  - for a given data set of points \(S = \{ (x_1, y_1), \ldots, (x_m, y_m) \}\)
  - select a point \(z\)
  - assume (local) approximation in the form
    \[ L_{XY}(x) = \alpha_1 + \alpha_2(x - z) + \ldots + \alpha_m(x - z)^{m-1} \]
  - assume weights for residuum, eg.
    \[ w_i = \frac{1}{(x_i - z)^2 + \varepsilon} \]
  - minimize the square of residuum weighted norm
    \[ ||r||^2 = r^T W r \]
  - with respect to \(a = [\alpha_1, \alpha_2, \ldots, \alpha_m]\)
  - by solution of the following SLAE \(A^T W A a = A^T W y\)
    where \(A\) is the Vandermonde matrix, \(A = A(z), W = W(z), a = a(z)\)
    \[ \varphi(z) = L_{XY}(z) = \alpha_1, \quad \text{note: } \frac{d \varphi}{dz} \neq \frac{d L_{XY}}{dx} \big|_{x=z} \]

- A basis function constructed by MLS approximation for the 3rd out of 6 nodes

![Figure 3: A 1D basis function constructed by MLS technique.](image)

4. Homework

- For functions shown in Fig.2 compute corresponding entries of the element vectors assuming a constant distributed loading.
- Discuss the possible types of coefficients of the SLAE (30).
- Compute the value at \(z = 1.5\) of the MLS basis function associated with second node. Assume 3 nodes at 0,1,2.
7 Introduction to nonlinear analysis

1. Linear problem assumptions

- Linear constitutive law (e.g. elasticity)
- Infinitesimally small displacements
- Small strains
- Nature of b.c. remains unchanged
- Gaps (debonding) do not appear during deformations

2. Typical nonlinear problems

- Nonlinear constitutive relation (e.g. elastic-plastic or visco-elastic material)
- Large displacements, small strains
- Large displacements, large strains
- Change of b.c. during deformation (contact, free boundary)
- Debonding of composite material components

3. Examples of applications and profits from nonlinear modeling

- Response of structures to extreme events
- Failures and deformations of soils
- Residual stress determination
- Structure life-time prediction
- Validation of linear models

4. MULTIPLE solutions or NO solution may exist
5. 1D example - large strains

- Strong formulation

\[
\begin{align*}
\frac{d\sigma}{dx} &= q(x) \quad \forall x \in (0, l) \\
\sigma &\approx E\varepsilon \quad \forall x \in (0, l) \\
\varepsilon &= \frac{du}{dx} + \frac{1}{2} \left( \frac{du}{dx} \right)^2 \quad \forall x \in (0, l) \\
u(0) &= 0 \\
A\sigma(l)u(l) &= P
\end{align*}
\quad (31)
\]

- Weak formulation

Find \( u(x) \in H_1 \), such that \( u(0) = 0 \) and

\[
A \int_0^l v'\sigma(u) \, dx = \int_0^l vq \, dx + v(l)P \quad \forall v \in V_0
\quad (32)
\]

- Newton-Raphson linearization

Given \( u_n \), find \( \psi(x) \in H_1 \), such that \( \psi(0) = 0 \) and

\[
A \int_0^l v'\frac{d\sigma}{du'} \frac{d\varepsilon}{du'}_{u=u_n} \psi' \, dx = \int_0^l vq \, dx + v(l)P - A \int_0^l v'\sigma(u'_n) \, dx \quad \forall v \in V_0
\quad (33)
\]

then \( u_{n+1} = u_n + \psi \)

- Element tangent matrix and vector for linear shape functions and \( q = \text{const} \)

\[
K_{el} = \frac{AE}{h}(1 + u'_h) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad P_{el} = \frac{qh}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - AE\varepsilon_h \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\quad (34)
\]

- Results for \( AE = 100 \), \( l = 1 \) \( q = 30 \), \( P = -10 \)

![Graph](image.png)

**Figure 4:** Displacement along the axis and \( u(l) \) versus load level for nonlinear (in red) and linear (in green) models. Maximum strain is equal to about 20\%.
6. Elastic - plastic deformations

- Strong formulation in 1D

\[
\begin{align*}
\frac{d\sigma}{dx} &= q(x) \quad \forall x \in (0, l) \\
\sigma &= E(\varepsilon - \varepsilon^p) \quad \forall x \in (0, l) \\
\varepsilon &= \frac{du}{dx} \quad \forall x \in (0, l) \\
u(0) &= 0 \\
A\sigma(l)n(l) &= P
\end{align*}
\] (35)

- Associated flow rule

\[
\dot{\varepsilon}^p = \gamma \frac{\partial \Phi}{\partial \sigma}
\] (36)

\[\Phi(\sigma, \varepsilon^p) = |\sigma - H\varepsilon^p| - \sigma_Y \leq 0\] - Huber-Mises-Hencky (von Mises) yield surface
\[\Phi \leq 0, \gamma \geq 0, \gamma \Phi = 0, \dot{\gamma} \Phi = 0\] - Kuhn-Tucker loading-unloading conditions
\[\dot{\sigma} = E_T \dot{\varepsilon} \quad E_T = E \text{ or } E_T = \frac{E_H}{E + H}\]

- Weak formulation

Find \( u(x) \in H_1 \), such that \( u(0) = 0 \) and

\[
A \int_0^l v'\sigma(u) \, dx = \int_0^l vq \, dx + v(l)P \quad \forall v \in V_0
\] (37)

- Newton-Raphson linearization

Given \( u^n_k \), \( \varepsilon^{p(n+1)} = \varepsilon^{p(n)} \), find \( \psi(x) \in H_1 \), such that \( \psi(0) = 0 \) and

\[
A \int_0^l v' E_T(u^{(n+1)}_k) \psi' \, dx = \int_0^l vq \, dx + v(l)P - A \int_0^l v'\sigma^{(n+1)}_k \, dx \quad \forall v \in V_0
\] (38)

then \( u_{n+1} = u_n + \psi \)

- Radial return for every Gauss point

\[
\sigma^{tr} = E(\varepsilon_n - \varepsilon^p_n + \Delta \varepsilon), \quad \text{If } \Phi(\sigma^{tr}) > 0 \text{ then } \Delta \gamma = \frac{\Phi(\sigma^{tr})}{E + H}
\]

- Results for a composite bar (two materials)

![Diagram of displacement versus load comparison]

Figure 5: Displacement versus load. Comparison of analytical and numerical solutions.

- Homework

  - Perform one step of the Newton-Rahpson method for problem (33).
  - Perform one step of the Newton-Rahpson method for problem (38).